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ON AN ITERATIVE PROCESS IN THE SOLUTION OF THE REGULAR N-BODY PROBLEM

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INTRODUCTION

This paper applies to the n-body problem of the general iterative process generated in ref [6] for the solution of any system of differential equations with analytic properties. In a quite specialized form, and in a somewhat different context, a comparable method is given in ref [4] for the solution of the problem.

If the solution of a system of differential equations is identified with the invariant of a complete metric space under a contraction mapping of the space into itself, [3 pp. 43-44] then the process becomes a method of approximating (successively) to the solution. The first part of the paper thus evolves this formal methodology in the solution of the problem. The iterative solution is expressed in two forms. The first statement of the iterative solution is recursive and the second is linear and irreducible.

The latter gives the iterative solution as a function of an arbitrary function which is the initial element of the iterative sequence. Three items are involved in the linear iterative formulation: an arbitrary function, coefficients and an error function. For the linear expression to be well defined, the arbitrary function must be specified, the coefficients must be given in terms of the initial conditions of the motion of the n-bodies and the error function must be characterized. These problems are resolved in the first part.

The analytical phases of the problem is discussed in the second part. Here the basic question of the connection between the elements of the iterative sequence and the actual solution is answered. The matter of the region of convergence of the series representing the actual solution, is formulated in two forms and the rate of convergence of the elements of the iterative sequence relative to (what is called) the partial sum sequence is given a measure. In the final part, a summary of the paper with a very brief directive in numerical determination, is given.

I. THE ITERATIVE METHODOLOGY

This part deals with the formal iterative process in the solution of the n-body problem. However, in its generation a serial a serial aspect will be involved, so that the process may be described as a pseudo iterative mode in the determination of a sequence of successive approximations to the solution. In turn this sequence will be transformed into a linear expression where any element will depend on the initial arbitrary element of the sequence.

Since this part of the paper depends, in a measure, on the two references [4], [6], we will repeat a few items, for the sake of completeness, of the methodology contained there. The analytical justification of the process will be considered in the part that follows.

<u>Related Systems of Differential Equations</u>: Three related systems [6] of differential equations are generated in this section. The initial system of the n-body problem is given by the equations of motion:

(1.1) $\ddot{x}^{ih} = \sum_{j} H_j S^{ij} X^{ijh}; \; H_j = Gm_j; \; i, j = 1, 2, \ldots, n; \; h = 1, 2, 3; \; i \neq j$

(1.2) x^i : Position vector of the i th point mass, m_i.

(1.3) x^{ih} : Scalar components of x^i relative to a Cartisian inertial coordinate system.

(1.4) $X^{ij} = x^j - x^i$: The relative position vector from the i to the j particle.

(1.5) $X^{ijh} = x^{jh} - x^{ih}$: Scalar components of the relative vector X^{ij}.

(1.6) $R^{ij} = |X^{ij}|^2 = \sum_k (X^{ijk})^2 = \sum_k (x^{jk} - x^{ik})^2$: The square of the magnitude of the relative vector X^{ij}.

(1.7) $S^{ij} = (R^{ij})^{-\frac{3}{2}}$; A scalar quantity.

To the original system (1.1) two other systems are added:

(1.8) $$\ddot{\bar{x}}_\alpha^{ik} = \sum_j H_j S_\alpha^{ij} X_\alpha^{ijk}, \quad \alpha = 1, 2, 3, \cdots$$

(1.9) $$\ddot{x}_\alpha^{ik} = \sum_j H_j S_{\alpha-1}^{ij} X_{\alpha-1}^{ijk}$$

The second system (1.8) is derived from (1.1) by the addition of subscripts. These subscripts originate from the partial sums formed from the serial solution

(1.10) $$x^{ik}(t) = \sum_{k=0}^{\infty} \xi_k^{ik} t^k$$

of the system (1.1), namely

(1.10)$_1$ $$\bar{x}_\alpha^{ik} = \sum_{k=0}^{\alpha+b} \xi_k^{ik} t^k, \quad \alpha = 1, 2, \cdots$$; b a fixed integer > 0

The third system (1.9) is derived, in form, from (1.8), and may be called the iterative system of (1.1). It is manifest that the three systems are distinct at least formally. The first system consists of 3 n equations, whereas the other two systems are given by $3n \times \alpha$, $\alpha = 1, 2, \cdots$ expressions.

To give fully specialized meaning to (1.8), (1.9) we define the functions S and X of (1.1) and thus are able to express these same symbols with subscripts, by partial sums, namely

(1.11) (a) $R^{ij}(t) = \sum_{k=0}^{\infty} \rho_k^{ij} t^k$, (b) $S^{ij}(t) = \sum_{k=0}^{\infty} \sigma_k^{ij} t^k$

(c) $X^{ijk}(t) = \sum_{k=0}^{\infty} \eta_k^{ijk} t^k$, $\eta_k^{ijk} = \xi_k^{jk} - \xi_k^{ik}$

Let the partial sums of (1.10) and (1.11) define the symbol $\bar{x}_\alpha$ as given by (1.10), and the symbols R_α, S_α, X_α, by the expressions

(1.12) (a) $R_\alpha^{ij} = \sum_{k=0}^{\alpha+b} \rho_k^{ij} t^k$, (b) $S_\alpha^{ij} = \sum_{k=0}^{\alpha+b} \sigma_k^{ij} t^k$

(c) $X_\alpha^{ijh} = \sum_{k=0}^{\alpha+b} \eta_k^{ijh} t^k$, $\alpha = 1, 2, 3, \ldots$, $b > 0$ an assigned integer.

The symbol $\bar{X}$ though given by (1.10), is not well defined. We obviate this lack by connecting X and $\bar{X}$ by the specification that

(1.13) $\bar{X}_\alpha^{ih}(t) = X_\alpha^{ih}(t) + \bar{\epsilon}_\alpha^{ih}(t)$, $\alpha = 1, 2, 3, \ldots$

On first glance it would appear that $\bar{X}$ is still ill-defined since the symbol $\bar{\epsilon}$ has no meaning attached to it. However, it will be shown in the sequel (as it has been shown in [4] (in a different context) that

(1.14) $\lim_{\alpha \to \infty} \bar{\epsilon}_\alpha^{ih}(t) = 0$ for any i, h and some specified t region.

<u>The Non-recursive Form of the Iterative Solution</u>: To generate a non-recursive form of the iterative solution X_α^{ih} we first unfold its recursive form [4]. To do so substitute the right members of (1.8) and (1.9) in expression (1.13). This leads to the statement,

(2.1) $$\sum_j H_j S_\alpha^{ij} X_\alpha^{ijh} = \sum_j H_j S_{\alpha-1}^{ij} X_{\alpha-1}^{ijh} + \bar{\epsilon}_\alpha^{ih}$$

In view of definition (1.4), equation (2.1) becomes

$$\sum_j H_j S_\alpha^{ij}(x_\alpha^{jh} - x_\alpha^{ih}) = \sum_j H_j S_{\alpha-1}^{ij}(x_{\alpha-1}^{jh} - x_{\alpha-1}^{ih}) + \bar{\epsilon}_\alpha^{ih}$$

By solving this equation for x_α^{ih}, we generate the recursive form

(2.2) $$x_\alpha^{ih} = \frac{A_{\alpha-1}^i}{A_\alpha^i} x_{\alpha-1}^{ih} + \frac{C_\alpha^{ih}}{A_\alpha^i} - \epsilon_\alpha'^{ih} , \quad \alpha = 1, 2, 3$$

where the symbols A, C, ϵ' are defined as

(2.3) $$A_\alpha^i = \sum_j H_j S_\alpha^{ij}, \quad C_\alpha^{ih} = \sum_j H_j (S_\alpha^{ij} x_\alpha^{jh} - S_{\alpha-1}^{ij} x_{\alpha-1}^{jh}), \quad \epsilon_\alpha'^{ih} = \frac{\bar{\epsilon}_\alpha^{ih}}{A_\alpha^i}$$

The recursive sequence of successive approximations

(2.4) $$(x_0^{ih},\ x_1^{ih},\ x_2^{ih}, ---)$$

to the actual solution x^{ih} of (1.1) is generated by (2.2). This ordered set of solutions is formally valid, provided the coefficients can be expressed in terms of known or computable functions for any α and where $\bar{\epsilon}_\alpha^{ih}$ is well defined. For the latter entity a measure will be derived, in the sequel, through the realization that the actual solution and the sequence of iterative solutions x_α^{ih} are embedded in a complete metric space for which a contraction mapping is permissable such that the invariant of this mapped space into itself is the solution of (1.1) and which is the limit of the sequence of functions (2.4).

If it is assumed, for the time being, that the coefficients of (2.2) are well defined then it is manifest that each element in the sequence (2.4) is ultimately a function of the initial element x_0^{ih}. This function however is as yet undefined (arbitrary). To show the explicit dependence of the iterative solution x_α^{ih}, for any $\alpha > 0$ on the arbitrary function x_0^{ih}, the recursive formula (2.2) is reduced to the non-recursive form

(2.5) $$x_\alpha^{ih} = \left(\frac{A_0^i}{A_\alpha^i}\right) x_0^{ih} + \left(\frac{\sum_{q=0}^{\alpha-1} C_{\alpha-q}^{ih}}{A_\alpha^i} - \epsilon_\alpha^{ih}\right); \quad \epsilon_\alpha^{ih} = \frac{\sum_{q=0}^{\alpha-1} \bar{\epsilon}_{\alpha-q}^{ih}}{A_\alpha^i}; \quad \alpha = 1, 2, ---$$

and where it will be shown that

(2.6) $$\lim_{\alpha\to\infty} \epsilon_\alpha^{ih} = 0$$ for any i, h.

The formula (2.5) may be verified by mathematical induction.

The iterative elements $x_\alpha^{ih}(t)$ in the sequence (2.4) are now expressed by (2.5) as a linear function of the initial arbitrary element $x_0^{ih}(t)$ of the sequence. The expression of the solution as a recursive form (2.2) has been reduced to an irreducible linear form (2.5)

The linear statement (2.5) unfolds two problems whose resolution is necessary for a complete formal iterative representation of the actual solution

$$X^{ik}(t) = \lim_{\alpha \to \infty} X_\alpha^{ik}(t) \tag{2.7}$$

of the n-body problem. The first problem is to choose some appropriate scheme (natural to the context), of the many possible modes, for the selection of some one class of functions with regular properties over a region to replace the arbitrary function. This class should be an ordered set such that any function in this chosen sequence is a closer approximation to the actual solution $X^{ik}(t)$ than any of its predecessors. Secondly, it will have to be shown that the $\lim_{\alpha \to \infty} \epsilon_\alpha^{ik} = 0$. A third problem, contingent on the second, is involved, namely to show that ϵ_α^{ik} for a finite α is a measure of the error of the iterative solution X_α^{ik} relative to the actual solution X^{ik}. These three problems will be resolved in the succeeding part of the paper.

The Recursive Formulation of the Coefficients of the Linear Form: The first necessary step in an explicit statement of (2.5), is to give defined expression to the coefficients A and C. Two modes of expression are open, namely a recursive [4] and a non-recursive one. We deal with the former first.

The substitution of (1.12-b) in the definition (2.3) leads to the expression

$$A_\alpha^i = \sum_j \sum_{k=0}^{\alpha+b} H_j \sigma_k^{ij} t^k , \qquad \text{b a fixed integer} > 0. \tag{3.1}$$

By means of (2.3) (1.12-b) and (1.10) we get

$$C_\alpha^{ik} = \sum_j H_j \Big[\sum_{k=0}^{\alpha+b} \sigma_k^{ij} t^k \sum_{l=0}^{\alpha+1} \xi_l^{jk} t^l - \sum_{k=0}^{\alpha+b+1} \sigma_k^{ij} t^k \sum_{l=0}^{\alpha+b-1} \xi_l^{jk} t^l ,$$

which expression turns into

$$C_\alpha^{ik} = \sum_j H_j \xi_{\alpha+b}^{jk} \sigma_{\alpha+b}^{ij} t^{2(\alpha+b)} \tag{3.2}$$

We note that the serial coefficients ξ and σ must initially be formulated in recursive form in order to gain defined statements for A and C

The serial coefficients ρ, σ and ξ are given in reference [5] in the following form:

$$(3.3)\quad \rho_k^{ij} = \sum_k \sum_{\nu=0}^{k} \eta_{k-\nu}^{ijh} \eta_{\nu}^{ijh}, \quad \eta_\gamma^{ijh} = \xi_\gamma^{jh} - \xi_\gamma^{ih}, \quad k = 0,1,2,\dots, \quad i \neq j$$

$$(3.4)\quad \sigma_k^{ij} = K_k \sum_{\omega=1}^{k} (2k+\omega)\, \sigma_{k-\omega}^{ij}\, \rho_\omega^{ij}; \quad K_k = -(2k\rho)^{-1}, \quad k = 1,2,\dots$$

$$(3.5)\quad \xi_{k+1}^{ih} = (k^2+k)^{-1} \sum_j \sum_{\nu=0}^{k-1} H_j\, \sigma_{k-\nu-1}^{ij}\, \eta_\nu^{ijh}, \quad k > 0$$

Slight consideration of the formulae (3.3),---, (3.5) will show that by cyclical use of these expressions the values of ρ_k^{ij}, σ_k^{ij} and ξ_k^{ih} may be attained for any value of k and that each of these coefficients are ultimately dependent on the prior knowledge of ξ_0^{ih} and ξ_1^{ih} By means of the identities

$$(3.6)\quad \xi_0^{ih} = \left[x^{ih}(t)\right]_{t=0}, \quad \xi_1^{ih} = \left[\dot{x}^{ih}(t)\right]_{t=0},$$

the $6n$ boundary conditions for the n-body problem are specified. So that the coefficients are ultimately expressed as functions of the boundary conditions.

The coefficients A_α^{i} and C_α^{ih} of the linear solution (2.5) thus become, in view of (3.3),---, (3.6), well defined functions for any α and a given $b > 0$. The three recursive formulae may then be used either in the actual solution $x^{ih}(t)$ of (1.1) through the serial representation (1.10), or more significantly in the iterative solution $\dot{x}_\alpha^{ih}(t)$ given by (2.5).

<u>The Non-Recursive formulation of the Coefficients of the Linear Form:</u>

The definition of the coefficients A and C of the linear iterative solution (2.5) were given as functions of the three serial coefficients ρ, σ, ξ in the three recursive formulae (3.3),---, (3.5). A more potent formulation is now given for these three quantities in terms of the single coefficient ξ where ξ itself is a function of the $6n$ initial conditions.

The second type of formulation as an irreducible recursive expression, is given in ref [5] . The basic statement with its definitions is written as follows:

$$(4.1)\quad \xi^{ih}_{k+2} = \sum_{j=1}^{n}\sum_{p=0}^{k}\sum_{q=0}^{P(p)}(k-p)!\,H_j\,\frac{G}{H}\left|\begin{matrix} i\,j\,h \\ k\,p\,q\end{matrix}\right|_{t=0}, \quad k=0,1,2,---$$

$$(4.2)\quad \left|\begin{matrix} i\,j\,h \\ k\,p\,q\end{matrix}\right|_{t=0} = \eta^{ijh}_{k-p}\left(R^{ij}\right)^{-\frac{2p+3}{2}}_{t=0}\prod_{f=0}^{p}\left[\sum_{s=1}^{3}\sum_{\ell=0}^{[\frac{f}{2}]}(f,\ell)\,f!\,\eta^{ijs}_{\ell}\,\eta^{ijs}_{f-\ell}\right]^{\alpha^{p}_{fq}}$$

$$(4.3)\quad G \equiv G(k,p,\alpha_0) = \frac{2}{\Gamma(\gamma)}F(a;b;c,1)F(\alpha,\beta;\gamma;1);\ H \equiv H(\alpha^{p}_{fq},f) = \prod_{f=1}^{p}\Gamma(\alpha^{p}_{fq+1}+1)\left[\Gamma(f+1)\right]^{\alpha^{p}_{fq}}$$

$$(4.4)\quad a = \alpha^{k}_{0q}-k-p-\tfrac{3}{2},\ b=-p,\ c=-2p+\alpha^{p}_{0q}-\tfrac{1}{2};\ \alpha=-2p+\alpha^{p}_{0q}-1,\ \beta=k+\tfrac{5}{2},\ \gamma=-2p+k+\alpha^{p}_{0q}+2$$

$$(4.5)\quad \sum_{f=1}^{p} f\,\alpha^{p}_{fq}=p;\ \alpha^{p}_{0q}=p-\sum_{f=1}^{p}\alpha^{p}_{fq};\ \alpha^{0}_{00}\equiv 0,\ p=1,2,---;\ q=1,2,---,P(p);$$

$$P(p) = \sum_{e}(-1)^{e+1}P\left[p-\tfrac{1}{2}(3e^2\pm e)\right], \text{(partition function)};$$

$[\frac{f}{2}]$ = largest integer not exceeding $\frac{f}{2}$; $(f,\ell)=2$ or 1 according as $\ell<\frac{f}{2}$ or $\ell=\frac{f}{2}$.

To evaluate the coefficient σ^{ij}_{k} of the series (1.11b) use is made of the formula (3.8) of reference [5] This is given as

$$(4.6)\quad S^{ij}_{k} = \left(R^{ij}\right)^{m-k}\sum_{q=1}^{P(k)}D(m,\bar{k})\,K(k,\alpha^{k}_{fq})\prod_{f=0}^{k}\left(R^{ij}_{f}\right)^{\alpha^{k}_{fq}},$$

$$S^{ij}_{k} \equiv \frac{d^k S^{ij}}{dt^k},\quad R^{ij}_{f} \equiv \frac{d^f R^{ij}}{dt^f},\quad k=0,1,2,---;\ m=-\frac{3}{2}$$

$$(4.7)\quad D(m,\bar{k}) = m(m-1)---(m-\bar{k}),\quad \bar{k} = k-\alpha^{k}_{0q}$$

$$(4.8)\quad K(k,\alpha^{p}_{fq}) = \frac{k!}{\prod_{f=0}^{k}(\alpha^{k}_{fq})!\,(f!)^{\alpha^{k}_{fq}}}$$

and where the remaining symbols in (4.6) are defined by (4.2),---(4.5).

Since $S^{ij} = \sum_{k=0}^{\infty} \sigma_k^{ij} t^k$ and $R^{ij} = \sum_{k=0}^{\infty} \rho_k^{ij} t^k$,

it follows that

$$(4.9) \quad \frac{(S_k^{ij})_{t=0}}{k!} = \sigma_k^{ij}, \quad \frac{(R_f^{ij})_{t=0}}{f!} = \rho_f^{ij}$$

It further follows, from (4.5), that

$$(4.10) \quad \sigma_k^{ij} = \frac{(R^{ij})_{t=0}^{m-k}}{k!} \sum_{g=1}^{P(k)} DK(f!\rho_f^{ij})^{\alpha_{fg}^k}, \quad k=0,1,2,---; \ m=-\frac{3}{2}.$$

The expression for ρ_f^{ij} in view of (3.3), is given as

$$(4.11) \quad \rho_f^{ij} = \sum_{h=1}^{3} \sum_{v=0}^{f} \eta_{f-v}^{ijh} \eta_v^{ijh}; \quad \eta_s^{ijh} = \xi_s^{jh} - \xi_s^{ih}.$$

The first set of recursive equations (3.3), (3.4), (3.5) for the serial coefficients ρ, σ, ξ are in the form $\rho = \rho(\xi), \sigma = \sigma(\rho, \xi)$, $\xi = \xi(\sigma, \xi)$. These have now been transformed into irreducible recursive expressions in terms of ξ alone, namely $\rho = \rho(\xi)$, $\sigma = \sigma(\xi)$, $\xi = \xi(\xi)$ in the equations (4.11), (4.10) and (4.1). These in turn are readily seen to be functions of the $6n$ boundary conditions,

$$(4.12) \quad [x^{ih}(t)]_{t=0} = \xi_0^{ih}, \quad [\dot{x}^{ih}(t)]_{t=0} = \xi_1^{ih}, \quad i=1,2,---,n, \ h=1,2,3.$$

The coefficients A_α^i and C_α^{ih}, as given by (3.1) and (3.2), again become more explicitly formulated by the alternative process given by the irreducible recursive statements (4.1), (4.9), (4.10). We thus have at our disposal two processes in the formulation of the coefficients A and C of the linear iterative formula (2.5).

II THE ANALYTICAL PHASE

The formal phase of the iterative process in the solution of the n-body problem is embodied in the iterative formula (2.5) which has the properties that it is linear, non-recursive and irreducible. Associated with this expression are the two modes of determining the coefficients of (2.5).

The basic problem still remains and that is to show the analytical validity of the iterative expression, namely that

$$\lim_{\alpha\to\infty} X_\alpha^{ih}(t) = X^{ih}(t), \ t \in R \tag{5.0}$$

where, of course, $X^{ih}(t)$ is the actual solution of the system (1.1) and $X_\alpha^{ih}(t)$, $\alpha = 1, 2, ---$ is an element in the sequence $(X_0^{ih}, X_1^{ih}, X_2^{ih}, ---)$ of solutions of the iterative system (1.9) and where R is a complex region on which the right member of (1.1) is analytic for any i, j and h.

In conjunction with this basic problem are the secondary ones: to determine the region R of convergence of the power series expressing the actual solution; to find an appropriate expression for the arbitrary function $X_0^{ih}(t)$ in the iterative sequence of solutions $(X_0^{ih}, X_1^{ih}, ---)$ and to find some measure for the rapidity of convergence of the sequence relative to the actual solution.

The Radius of Convergence for the Actual Solution: Since the right member of (1.1) is analytic in the neighborhood of the origin and this combined with the specifications of the boundary conditions allows us to represent the solution by the power series,

$$X^{ih}(t) = \sum_{k=0}^{\infty} \xi_k^{ih} t^k \tag{5.1}$$

converging in the neighborhood of the origin. Our first concern then is to specify the region R of convergence in the complex plane of the representation (5.1). Two regions are specified in this section, one a minimal and the other a maximal region with their corresponding radii.

(1) The former is given in reference [1] . It is there specified that the series (5.1) converges for all real value of t in the interval given by

$$(5.2)_0 \qquad |t| < \frac{a}{c(n+1)}$$

where the following definitions and conditions are fulfilled by a and c:

$$(5.2)_1 \qquad a = \frac{1}{2}\theta \min_{i,j} a_{ij}\ , \quad 1 \le i \le n\ , \ 1 \le j \le n, \ i \ne j\ , \ 0 < \theta < 1$$

$$(5.2)_2 \qquad a_{ij} = \max_{1 \le p \le 3} |x^{jp} - x^{ip}|_{t=0}$$

$$(5.2)_3 \qquad c = \max_{k,i} (c^{ik}, d^{ik})$$

$$(5.2)_4 \qquad c^{ik} = \sum_{j=1}^{n} \frac{H_j(1+\theta)}{a_{ij}^2(1-\theta)^3}$$

$$(5.2)_5 \qquad d^{ik} = \max_{i,k} |x^{ik}(t)|_{t=0} + a$$

(2) The maximum finite region of convergence of the series (5.1) is discussed in reference [5] . We briefly recapitulate the essential items.

Since it has been assumed that the motion of the system of n-bodies is to be regular, namely that no collisions or other real discontinuities exist in the motion for all finite time, it follows that any divergence of the series (5.1) is a consequence of only complex singularities that exist in a finite region of the complex plane. The real axis must be void of any singularities.

In order to generate a time series which represents the solution for all values of the real finite time and so corresponds to the assumed regular (analytic) motion of the system, the following Poincare transformation [7, pp. 423-4] is used,

$$t = \frac{2\bar{h}}{\pi} \ln \frac{1+\tau}{1-\tau} \tag{5.3}$$

where t and τ are two complex variables and where $\bar{h}$ specifies the distance that the nearest singularity of the solution is from the real axis of the t plane. It is readily observed that the mapping (5.3) transforms "the interior of the circle $|\tau|=1$ on the τ plane into a band which extends to a distance $\bar{h}$ on either side of the real axis in the t plane," [7, p. 423] . It follows that if a series in τ converges for all τ in the region $|\tau|<1$, the transformed series will converge for all t on the real interval $-\infty < t < +\infty$

The series (5.1) as given in ref. [5] takes the form

$$X^{ih}(t) = \sum_{r=0}^{\infty} \xi_r^{ih} t^r = X_0^{ih}(0) + X_1^{ih}(0) + \sum_j \sum_{r=0}^{\infty} \frac{(\gamma_r^{ijh})_{t=0}}{(r+2)!} t^{r+2} \tag{5.4}$$

where $\gamma_r^{ijh}(t)$ is defined by equation (2.5) of the reference. Since it is known, as stated, that the solution is analytic in the neighborhood of $t = 0$, the series (5.4) converges in that neighborhood. So that the Cauchy limit ratio is valid. Thus with the use of the transformation (5.3) we generate the valid inequality,

$$\lim_{r\to\infty} \left| \frac{\sum_j \frac{H_j \gamma_{r+1}^{ijh}}{(r+3)!}}{\sum_j \frac{H_j \gamma_r^{ijh}}{(r+2)!}} \right|_{t=0} \left| \frac{\left(\frac{2\bar{h}}{\pi}\right)^{r+3} \left(\ln \frac{1+\tau}{1-\tau}\right)^{r+3}}{\left(\frac{2\bar{h}}{\pi}\right)^{r+2} \left(\ln \frac{1+\tau}{1-\tau}\right)^{r+2}} \right| < 1.$$

With the use of formula (1.10) ref. [5] namely $\gamma_{r+1}^{ijh} = \gamma_r^{ijh} \cdot \gamma_1^{ijh}$, the above expression is reduced to the form

$$\lim_{r\to\infty} \left(\frac{1}{r+3}\right) \left| \gamma_1^{ijh} \right|_{t=0} \frac{2\bar{h}}{\pi} \left| \ln \frac{1+\tau}{1-\tau} \right| < 1 ,$$

where $|y_1^{i\bar{j}h}|_{t=0}$ is the least value of the set $|y_1^{ijh}|_{t=0}$ for $j=\bar{j}$

In reference [5] it is shown that

$$|T| < \left|\frac{1+T}{1-T}\right| < e^{\frac{1}{\varphi}}, \quad \varphi = \frac{2h\,|y_1^{i\bar{j}h}|_{t=0}}{\pi \lim_{n\to\infty}(r+3)}$$

The quantity $\frac{1}{\varphi}=0$ provided $|y_1^{i\bar{j}h}|_{t=0}\neq 0$ and where $\bar{h}>0$ and finite. The expression

$$(5.5) \qquad |y_1^{i\bar{j}h}|_{t=0} = \left|\frac{dS^{i\bar{j}}}{dt}\cdot X^{i\bar{j}h} + S^{i\bar{j}}\frac{dX^{i\bar{j}h}}{dt}\right|_{t=0} \neq 0$$

implies that no movable singularities, given by $|y_1^{i\bar{j}h}|_{t=0}=0$, are allowable in the regular motion of the s ystem.

The discussion may thus be summarized by the

<u>Theorem</u>: The power series (5.4) represents the actual solution $X^{ih}(t)$ over the time interval $-\infty < t < +\infty$ provided that the movable singularities are accounted for, namely that condition (5.5) is satisfied at the initial state $(t=0)$ of the motion. The movable singularities [5] are given by the expressions

$$(5.6) \qquad (R^{ij})^{-1}_{t=0} = 0 \quad \text{or} \quad \left[\frac{X^{ijh}\,\frac{d}{dt}R^{ij}}{R^{ij}\,\frac{d}{dt}X^{ijh}}\right]_{t=0} = \frac{2}{3}$$

<u>The Limit of the Sequence of Iterative Solutions</u>: Once the existence of the actual solution and its region of validity is formulated, the basic problem presents itself, namely what relation does the sequence of iterative solutions bear to the actual solution. Is the limit relation given by (6.0) analytically valid? The problem has been resolved in a topological mode in references [6] and [3] , and by classical means, and in somewhat different context but identical structure, in reference [4] .

(1) In the first two references it is shown that the sequence of iterative solutions of any system of first order differential equa-

tions with analytical properties may be subsumed as elements of a complete metric space which permits a contraction mapping of the space into itself. The space under this transformation has a unique invariant (satisfying an initial point) which is the limit of a sequence of elements of the space. This invariant is the unique solution of the system of differential equations and the sequence is an iterative one, generated by the contraction mapping in the form of a system of systems of integral equations.

All the ingredients exist on changing the system of $3n$ equations (1.1) of the second order to a system of $6n$ equations of the first order with the same $6n$ initial boundary conditions and analytic properties. We are thus able to transcribe, with slight formal changes, the extended existence theorem, given in reference [6] , to the form,

<u>Theorem:</u> The analytic function $X^{ik}(t)$ which is the limit of the sequence $(X_0^{ik}, X_1^{ik}, X_2^{ik}, \ldots)$ of iterative solutions X_α^{ik} of a system of differential equations

$$\frac{dX_\alpha^{ik}}{dt} = f^{ik}(t, \mathbb{X}_{\alpha-1}^{ijk}), \left\{ X_{\alpha-1}^{jk} - X_{\alpha-1}^{ik} = \mathbb{X}_{\alpha-1}^{ijk};\ i, j = 1, 2, \ldots, n,\ k = 1, 2, 3,\ i \neq j, \right.$$

is also the solution $X^{ik}(t)$ (if $X^{ik}(t)$ exists) of the system of equations

$$\frac{dX^{ik}}{dt} = f^{ik}(t, \mathbb{X}^{ijk}), \quad \{\mathbb{X}^{ijk} = X^{jk} - X^{ik}\}$$

satisfying the same boundary conditions, provided the initial arbitrary function X_0^{ik} of the sequence is defined to be analytic on the same region as $X^{ik}(t)$ and satisfies the same boundary conditions.

If then the expression (2.5) is transformed into the form

$$(6.1)\quad X^{ik}(t) = \lim_{\alpha\to\infty} X_\alpha^{ik}(t) = \lim_{\alpha\to\infty} \frac{A_0^i}{A_\alpha^i} X_0^{ik} + \lim_{\alpha\to\infty} \frac{\sum_{q=0}^{\alpha-1} C_{\alpha-q}^{ik}}{A_\alpha^i} - \lim_{\alpha\to\infty} \epsilon_\alpha^{ik}$$

then the theorem further tells us that the limits of the coefficients exist and are bounded. This is so, since it has been shown that the solution $X^{ik}(t)$ of (6.1) is analytic over a defined region. It will be verified by an independent classical consideration, that such is the case. We first give an expression for the arbitrary function.

(2) A glance at the statement (6.1) makes it manifest that the limit function $X^{ik}(t)$ remains undefined to the extent in which the arbitrary function $X_0^{ik}(t)$ is unspecified. Of the many possible ways of definition, that mode is considered which is closely associated with the specification of the coefficients.

Thus the partial sum (1.10),

$$(6.2)\qquad \bar{X}_{\alpha}^{ik} = \sum_{k=0}^{\alpha+b} \xi_k^{ik} t^k \quad , \qquad b>0, \text{ fixed integer},$$

of the series (1.10) is used. If $\alpha = 0$, define the arbitrary function by the expression

$$(6.3)\qquad X_0^{ik}(t) = \bar{X}_b^{ik}(t) = \sum_{k=0}^{b} \xi_k^{ik} t^k$$

It is manifest that the larger the value assigned to b the closer will the initial function $X_0^{ik}(t)$ (and therefore every element in the iterative sequence) be to the actual solution $X^{ik}(t)$. The now defined function, $X_0^{ik}(t)$, may readily be shown to fulfill the two conditions demanded by the theorem. Since it is a polynomial, it is analytic in any region for which the solution is analytic, and for $t=0$ the boundary conditions are satisfied.

(3) The independent classical proof of the validity of the form, given by (6.1), is developed in reference [4, p. 577] . The results are summarized in the following

<u>Theorem</u>: The limit of the sequence $(X_0^{ik}, X_1^{ik}, X_2^{ik}, ---)$ is given by the

equality

$$\lim_{\alpha\to\infty} X_\alpha^{ih}(t) = X^{ih}(t)$$

for any i, h and t in a defined region R; $X^{ih}(t)$ is the regular solution on R of the system of differential (1.1); $X_\alpha^{ih}(t)$ $(\alpha = 1, 2, ---)$ is the iterative solution given by formula (2.2).

In the proof of this version of the theorem [4] it is shown that the following limits exist and have the following values:

(6.4) $\lim_{\alpha\to\infty} \epsilon_\alpha^{ih}(t) = 0$; $\lim_{\alpha\to\infty} A_\alpha^i(t) \neq 0$; $\lim_{\alpha\to\infty} \sum_{q=0}^{\alpha-1} C_{\alpha-q}^{ih}$ is bounded for any i, h and all t in the defined region R. If to these facts we add the statement that the initial function $X_0^{ih}(t)$ as given by (6.3), is defined analytic on R, we are in a position to formulate two statements:

<u>Theorem:</u> The expression (6.1) may be written in the form

$$(6.5) \qquad X^{ih}(t) = \lim_{\alpha\to\infty} \frac{A_0^i}{A_\alpha^i} X_0^{ih} + \lim_{\alpha\to\infty} \frac{\sum_{q=0}^{\alpha-1} C_{\alpha-q}^{ih}}{A_\alpha^i}$$

where $A_\alpha^i(t) \neq 0$ and $\sum_{q=0}^{\alpha-1} C_{\alpha-q}^{ih}(t)$ is bounded for any i, h, and all $t \in R$

The second statement is contingent on the radius of convergence of the solution series. If the minimal radius of convergence is considered and as given by $(5.2)_0$, then we may formulate the following

<u>Theorem:</u> The terms in the right member of (6.5) are analytic on the circular region R of radius $|t| < \frac{a}{c(n+1)}$ where the quantities a and c are given by $(5.2)_1$, --- $(5.2)_5$.

If the maximal radius is given, then we may write the following

<u>Theorem:</u> The terms in the right member of (6.5) are analytic on the circular region R of radius $|t| < \infty$ provided that

$$|Y_1^{ijh}| \equiv \left(\frac{d}{dt} S^{ij} \cdot X^{ijh} + S^{ij} \frac{d}{dt} X^{ijh}\right)_{t=0} \neq 0$$

<u>The Measure of Convergence of the Iterative Solution</u>: The iterative and approximate sequences

(7.1) (a) $(x_0^{ik}, x_1^{ik}, x_2^{ik}, ---)$, (b) $(\bar{x}_\beta^{ik}, \bar{x}_{\beta+1}^{ik}, \bar{x}_{\beta+2}^{ik}, ---)$

have different rates of convergence to the limit solution $x^{ik}(t)$ of the system of differential equations (1.1). Our object in this section is to find a measure of the relative rate.

It was shown in ref. [6] that the above sequences, successive approximate and partial sum solutions of systems (1.9) and (1.8) respectively, are imbedded in a complete metric space under a contraction mapping of the space into itself. This space has an invariant under the mapping which is the solution $x^{ik}(t)$ of the original system (1.1), with the property that

(7.2) $\lim_{\alpha\to\infty} \bar{x}_{\beta+\alpha}^{ik}(t) = \lim_{\alpha\to\infty} x_\alpha^{ik}(t) = x^{ik}(t)$, $\beta > 0$ fixed integer; $t \in R$

The metric, ρ, of the space has by definition [3], the following properties:

(7.3) $\rho(x,y) = 0, x = y$; $\rho(x,y) = \rho(y,x)$; $\rho(x,z) \le \rho(x,y) + \rho(y,z)$,

where x, y, z are any elements of the space. Finally in ref. [6] we defined the distance (metric) of the element (or curve) $y_v(t)$ to the curve (or element) $y_w(t)$ of the space by the statement that

(7.4) $\rho(y_v, y_w) \equiv \rho_{v,w} = \max_t |y_v(t) - y_w(t)|$

In the above reference it was given that

(7.4)' $\rho_{\alpha,\alpha+r} \le Mh\, \rho_{\alpha-1,\alpha+r-1}$; $\alpha, r = 1, 2, ---$; $Mh < 1$

for any two elements $y_\alpha^{ik}(t)$ and $y_{\alpha+r}^{ik}(t)$ in the sequence (7.1-a) and where M is the largest in the set of upper bounds of the functions

$\sum_{j} S^{ij} X^{ijh}$, and $h = \frac{a}{c(n+1)}$ as given by $(5.2)_0$.

Expression (7.5) states that the maximum of the absolute difference between any two curves y_{α}^{ih} and $y_{\alpha+r}^{ih}$ in the sequence (7.1-a) is less than the maximum of the absolute difference of the corresponding immediate predecessors.

The difference of the error function ϵ for any two curves in the sequence (7.1-a) may be formulated, by the following considerations. By definition (7.4) and (2.5),

$$\rho_{\alpha,\alpha+r} = \max_{t,i,h} \left| X_{\alpha+r}^{ih} - X_{\alpha}^{ih} \right| = \max_{t,i,h} \left| D_{\alpha+r}^{ih} - D_{\alpha}^{ih} \right| + \left| \epsilon_{\alpha}^{ih} - \epsilon_{\alpha+r}^{ih} \right|$$

where

(7.5) $$D_{\gamma}^{ih} = \frac{A_0^i}{A_{\gamma}^i} X_0^{ih} + \frac{\sum_{q=0}^{\gamma-1} C_{\gamma-q}^{ih}}{A_{\gamma}^i}$$

Since

$$\max_{t,i,h} \left| \left(D_{\alpha+r}^{ih} - D_{\alpha}^{ih}\right) + \left(\epsilon_{\alpha}^{ih} - \epsilon_{\alpha+r}^{ih}\right) \right| \leq \max_{t,i,h} \left| D_{\alpha+r}^{ih} - D_{\alpha}^{ih} \right| + \max_{t,i,h} \left| \epsilon_{\alpha}^{ih} - \epsilon_{\alpha+r}^{ih} \right|$$

it follows that

(7.6) $$\max_{t,i,h} \left| \epsilon_{\alpha}^{ih} - \epsilon_{\alpha+r}^{ih} \right| \geq \rho_{\alpha,\alpha+r} - \max_{t,i,h} \left| D_{\alpha+r}^{ih} - D_{\alpha}^{ih} \right|$$

The maximum of the absolute difference of the errors of any two curves of the iterative sequence relative to the solution $y^{ih}(t)$ is given in terms of the elements of the sequence as determined by the linear formula (2.5).

To measure the relative rate of convergence of the two sequences, relations must be established between the elements in the two sequences. To do so we start with the basic definition of contraction mapping in a complete metric space, namely that $\rho_{m,n} \leq \alpha\, \rho_{m-1,n-1} < \rho_{m-1,n-1}, \alpha < 1$

Specifically, for the elements $\bar{X}_{\gamma}^{ih}, \bar{X}_{\gamma+u}^{ih}$ of the sequence (7.1-b), we write

(7.7) $\rho_{\gamma,\gamma+u} < \rho_{\gamma-1,\gamma+u-1}$, $\gamma = \beta + s$, β a fixed integer > 0 , $s = 0, 1, 2, \ldots$

The convention is established that a subscript β or γ will imply reference to the curves of the second sequence (7.1-b)

Expression (7.7) is given in terms of the elements as

$$\max_{t,i,h}\left|\bar{X}^{ih}_{\beta+s+u}-\bar{X}^{ih}_{\beta+s}\right|<\max_{t,i,h}\left|\bar{X}^{ih}_{\beta+s+u-1}-\bar{X}^{ih}_{\beta+s-1}\right|$$

In view of expression (6.2), we write

$$\max_{t,i,h}\left|\sum_{k=0}^{\beta+s+u}\xi_k^{ih}t^k-\sum_{k=0}^{\beta+s}\xi_k^{ih}t^k\right|<\max_{t,i,h}\left|\sum_{k=0}^{\beta+s+u-1}\xi_k^{ih}t^k-\sum_{k=0}^{\beta+s-1}\xi_k^{ih}t^k\right|$$

or

$$\max_{t,i,h}\left|\sum_{k=\beta+s+1}^{\beta+s+u}\xi_k^{ih}t^k\right|<\max_{t,i,h}\left|\sum_{k=\beta+s}^{\beta+s+u-1}\xi_k^{ih}t^k\right|$$

This leads to the statement

$$(7.8)\qquad \rho_{\gamma,\gamma+u}<\max_{t,i,h}\left|\sum_{k=\gamma}^{\gamma+u-1}\xi_k^{ih}t^k\right|$$

where

$$(7.8)'\quad (a)\ \rho_{\gamma,\gamma+u}=\max_{t,i,h}\left|\sum_{k=\gamma+1}^{\gamma+u}\xi_k^{ih}t^k\right|,\ (b)\ \rho_{\gamma-1,\gamma+u-1}=\max_{t,i,h}\left|\sum_{k=\gamma}^{\gamma+u-1}\xi_k^{ih}t^k\right|$$

Expression (7.7) or (7.8) specifies that the distance (or maximal difference) between any two functions, in the partial sum sequence (7.1-b), is less than the distance of the two corresponding immediate predecessors.

It is specified that the measurement of the metric is to start from the initial element common to both sequences, namely

$$(7.9)\qquad y_0^{ih}(t)=\bar{y}_\beta^{ih}(t)\quad,\qquad \beta \text{ a fixed integer} >0$$

as defined by (6.3). Consider the functions $y_\alpha^{ih}, y_{\alpha+r}^{ih}$ and $y_{\beta+s}^{ih}, y_{\beta+s+1}^{ih}$ of the two sequences. Then, in view of the triangular rule (7.3), for the metric, the connection between the elements of the two sequences are given by the inequalities

(7.10) $\rho_{0,\alpha+r} \leq \rho_{0,\beta+s} + \rho_{\beta+s,\alpha+r}$

(7.11) $\rho_{0,\beta+s+1} \leq \rho_{0,\alpha+r} + \rho_{\alpha+r,\beta+s+1}$

Expression (7.10) relates the metrics between the elements $\eta_0^{ih}, \eta_{\alpha+r}^{ih}$ of the first sequence and the elements $\bar{\eta}_\beta^{ih} = \eta_0^{ih}, \bar{\eta}_{\beta+s}^{ih}$ and $\bar{\eta}_{\beta+s}^{ih}, \eta_{\alpha+r}^{ih}$ of the second and first sequences. A similar situation holds for (7.11).

The inequality (7.10) leads to

(7.12) $\rho_{\beta+s,\alpha+r} \geqq \rho_{0,\alpha+r} - \rho_{0,\beta+s}$

provided the condition

(7.12)' $\rho_{0,\alpha+r} \geqq \rho_{0,\beta+s}$

is satisfied. Likewise (7.11) gives

(7.13) $\rho_{\alpha+r,\beta+s+1} \geqq \rho_{0,\beta+s+1} - \rho_{0,\alpha+r}$

provided

(7.13)' $\rho_{0,\beta+s+1} \geqq \rho_{0,\alpha+r}$

Conditions (7.12)' and (7.13)' are put into the form

(7.14) $\rho_{0,\beta+s} \leq \rho_{0,\alpha+r} \leq \rho_{0,\beta+s+1}$.

It is significant to note that if the condition (7.14) holds for the three elements $\bar{\eta}_{\beta+s}, \eta_{\alpha+r}, \bar{\eta}_{\beta+s+1}$, then if a mixed sequence of the two sets of elements are formed, the order of the three elements will be as given above. Furthermore the two elements $\bar{\eta}_{\beta+s}, \bar{\eta}_{\beta+s+1}$ of the second sequence (7.1-b) will have the shortest distance to $\eta_{\alpha+r}$ on either side, for any two elements of (7.1-b). If then we assume that the inequality relations (7.14) is fulfilled, we generate the order

$$\bar{\eta}_{\beta+s}, \eta_{\alpha+r}, \bar{\eta}_{\beta+s+1}.$$

This quantity s must simultaneously satisfy the inequality (7.12) and (7.13). Furthermore for $\bar{\eta}_{\beta+s}$ and $\bar{\eta}_{\beta+s+1}$ of the sequence to be nearest to $y_{\alpha+r}$ on either side, it must also fulfill the conditions (7.14). The quantity s thus indicates the number of terms in the interval of the second sequence from $\bar{y}_\beta = y_0$ to $\bar{y}_{\beta+s}$ required to cover the interval from y_0 to $y_{\alpha+r}$ of the iterative sequence. In this sense the coverage given by s , is a measure of the rate of convergence of the iterative relative to the partial sum sequence.

'The explicit' statements of (7.12), (7.13) and (7.14) are given by means of the definition of the metric (7.4) respectively as follows:

(7.15) $$\max_{t,i,h} \left|\bar{X}^{ih}_{\beta+s} - X^{ih}_{\alpha+r}\right| \geq \max_{t,i,h} \left|X^{ih}_0 - X^{ih}_{\alpha+r}\right| - \max_{t,i,h}\left|X^{ih}_0 - \bar{X}^{ih}_{\beta+s}\right|$$

(7.16) $$\max_{t,i,h} \left| X^{ih}_{\alpha+r} - \bar{X}^{ih}_{\beta+s+1}\right| \geq \max_{t,i,h}\left|X^{ih}_0 - \bar{X}^{ih}_{\beta+s+1}\right| - \max_{t,i,h}\left|X^{ih}_0 - X^{ih}_{\alpha+r}\right|$$

(7.17) $$\max_{t,i,h}\left|X^{ih}_0 - \bar{X}^{ih}_{\beta+s}\right| \leq \max_{t,i,h}\left|X^{ih}_0 - X^{ih}_{\alpha+r}\right| \leq \max_{t,i,h}\left|X^{ih}_0 - \bar{X}^{ih}_{\beta+s+1}\right|$$

The barred elements $\bar{X}$ are given by the series (6.4) and the unbarred elements X by the linear form (2.5)

A useful inequality may be derived from expressions (7.12) and (7.13). By adding these statements we get

$$\rho_{\beta+s,\alpha+r} + \rho_{\alpha+r,\beta+s+1} \geq \rho_{0,\beta+s+1} - \rho_{0,\beta+s}$$

Since $\rho_{\beta+s,\alpha+r} \geq \rho_{\beta+s+1,\alpha+r} = \rho_{\alpha+r,\beta+s+1}$ it follows that

(7.18) $$\rho_{\beta+s,\alpha+r} \geq \frac{\rho_{0,\beta+s+1} - \rho_{0,\beta+s}}{2}.$$

Expression (7.18) gives a measure of the mixed metric in terms of consecutive curves of the same sequence (7.1-b)

III THE NUMERICAL PROCESS AND SUMMARY

This part deals with a list of the formulae for numerical processes, brief specification of these processes and which at the same time gives a summary of pertinent results. The numbers attached to the formulae correspond to those in the body of the paper.

The formal iterative solution of the system of equations

(1.1) $\ddot{X}^{ih} = \sum_j H_j S^{ij} X^{ijh}$; $H_j = Gm_j$; $i, j = 1, 2, \ldots, n$; $h = 1, 2, 3$; $i \neq j$

is given by two forms,

(2.2) $X_\alpha^{ih} = \frac{A_{\alpha-1}^i}{A_\alpha^i} X_{\alpha-1}^{ih} + \frac{C_\alpha^{ih}}{A_\alpha^i} - \epsilon_\alpha^{ih}$, $\alpha = 1, 2, \ldots$ (recursive form)

(2.5) $X_\alpha^{ih} = \frac{A_0^i}{A_\alpha^i} X_0^{ih} + \frac{\sum_{q=0}^{\alpha-1} C_{\alpha-q}^{ih}}{A_\alpha^i} - \epsilon_\alpha^{ih}$, $\alpha = 1, 2, \ldots$ (irreducible linear form)

(2.6) $\lim_{\alpha \to \infty} \epsilon_\alpha^{ih} = 0$ for any i, h

The coefficients A and C may be computed by either one of two sets of formulae. The first set is given as follows:

(3.1) $A_\alpha^i = \sum_{j=1}^{n} \sum_{k=0}^{\alpha+b} H_j \sigma^{ij} t^k$, b a fixed integer > 0

(3.2) $C_\alpha^{ih} = \sum_j H_j \xi_{\alpha+b}^{jh} \sigma_{\alpha+b}^{ij} t^{2(\alpha+b)}$

(3.3) $\rho_k^{ij} = \sum_{h=1}^{3} \sum_{\nu=0}^{k} \eta_{k-\nu}^{ijh} \eta_\nu^{ijh}$; $\eta_\gamma^{ijh} = \xi_\gamma^{jh} - \xi_\gamma^{ih}$; $k = 0, 1, 2, \ldots$; $i \neq j$.

(3.4) $\sigma_k^{ij} = K_k \sum_{\omega=1}^{k} (2k+\omega) \sigma_{k-\omega}^{ij} \rho_\omega^{ij}$; $K_k = -(2k\rho_0)^{-1}$; $k = 1, 2, \ldots$

(3.5) $\xi_{k+1}^{ih} = (k^2+k)^{-1} \sum_j \sum_{\nu=0}^{k-1} H_j \sigma_{k-\nu-1}^{ij} \eta_\nu^{ijh}$, $k > 0$

(1.12) (a) $R_\alpha^{ij} = \sum_{k=0}^{\alpha+b} \rho_k^{ij} t^k$, (b) $S_\alpha^{ij} = \sum_{k=0}^{\alpha+b} \sigma_k^{ij} t^k$, (c) $X_\alpha^{ijh} = \sum_{k=0}^{\alpha+b} \eta_k^{ijh} t^k$, $\eta_k^{ijh} = \xi_k^{jh} - \xi_k^{ih}$.

(1.11) (a) $R^{ij} = \sum_{k=0}^{\infty} \rho_k^{ij} t^k$, (b) $S^{ij} = \sum_{k=0}^{\infty} \sigma_k^{ij} t^k$, (c) $X^{ijh} = \sum_{k=0}^{\infty} \eta_k^{ijh} t^k$, $\eta_k^{ijh} = \xi_k^{jh} - \xi_k^{ih}$.

(3.6) $\xi_0^{ih} = [X^{ih}(t)]_{t=0}$, $\xi_1^{ih} = [\dot{X}^{ih}(t)]_{t=0}$, initial conditions

By cyclical use of the recursive expressions (3.3),---(3.5) the values of ρ_k^{ij}, σ_k^{ij} and ξ_k^{ih} may be computed for any k Each of these coefficients are then ultimately expressed in terms of the initial conditions given by (3.6)

The second set of expressions in the numerical determination of A and C are given in irreducible recursive forms. These are listed in the text from statements (4.1) to (4.11). A brief description is in order. The coefficients ξ in (3.2) are given by equations (4.1) and the associated ones (4.2),---(4.5). For the coefficients σ the expressions (4.6) to (4.11) are used. These quantities reduce to functions of the initial conditions given by (4.12). In reference [5] a brief specification is given of a process in the numerical determination of the ξ's.

In what follows we are guided by the fact that

(5.6) $\lim\limits_{\alpha \to \infty} x_\alpha^{ih} = \lim\limits_{\alpha \to \infty} \bar{x}_\alpha^{ih} = x^{ih}(t)$, $t \in R$

where $x^{ih}(t)$, the solution of system (1.1) is given by the series

(1.10) $$x^{ih}(t) = \sum_{k=0}^{\infty} \xi_k^{ih} t^k,$$

where $\bar{x}_\alpha^{ih}$, the partial sum solution of

(1.8) $$\ddot{\bar{x}}_\alpha = \sum_j H_j S_\alpha^{ij} X_\alpha^{ijh}, \quad \alpha = 1, 2, 3, ---$$

has the form

(1.10)$_1$ $$\bar{x}_\alpha^{ih} = \sum_{k=0}^{\alpha+\beta} \xi_k^{ih} t^k, \; \alpha = 1, 2, --- ; \quad \beta \text{ a fixed integer} > 0$$

and described by the sequence (7.1-a) $(\bar{x}_\beta^{ih}, \bar{x}_{\beta+1}^{ih}, \bar{x}_{\beta+2}^{ih}, ---)$ and finally where x_α^{ih} , the iterative solution of the system

(1.9) $$\ddot{x}_\alpha^{ih} = \sum_j H_j S_{\alpha-1}^{ij} X_{\alpha-1}^{ijh},$$

is given by the recursive expression (2.2) or non-recursive linear form (2.5) and described by the sequence $(x_\beta^{ih}, x_{\beta+1}^{ih}, x_{\beta+2}^{ih}, ---)$

(7.1-b) $(\bar{X}^{ih}_{\beta}, \bar{X}^{ih}_{\beta+1}, \bar{X}^{ih}_{\beta+2}, ---)$. We defined the initial functions of the two sequences by (6.3), namely

(6.3) $$X_0^{ih}(t) = \bar{X}^{ih}_{\beta}(t) = \sum_{k=0}^{\beta} \xi^{ih}_k t^k, \qquad \beta \text{ a fixed integer} > 0$$

The radius of convergence of the series representation (1.10) of the actual solution $X^{ih}(t)$ of (1.1) is given in two forms, namely as minimal and maximal values. The former is given as

$(5.2)_0$ $$|t| < \frac{a}{c(n+1)}$$

$(5.2)_1$ $$a = \frac{1}{2}\theta \min_{i,j} a_{i,j}, \quad 1 \le i,j \le n, \ i \ne j, \quad 0 < \theta < 1$$

$(5.2)_2$ $$a_{i,j} = \max_{1 \le p \le 3} \left| X^{jp}_{(0)} - X^{ip}_{(0)} \right|; \quad X^{jp}_{(0)}, X^{ip}_{(0)} \quad \text{initial conditions}$$

$(5.2)_3$ $$c = \max_{h,i} (c^{ih}, d^{ih}), \quad 1 \le h \le 3$$

$(5.2)_4$, $(5.2)_5$ $$c^{ih} = \sum_{j=1}^{n} \frac{H_j(1+\theta)}{a^2_{ij}(1-\theta)^3}; \quad d^{ih} = \max_{i,h} \left| X^{ih}_{(0)} \right| + a$$

The maximal range for the actual solution (1.10) takes the form

$$|t| < \infty, \text{ provided } (R^{ij})^{-1}_{t=0} \text{ or } \left| \frac{\mathbf{X}^{ijh} \frac{d}{dt} R^{ij}}{R^{ij} \frac{d}{dt} \mathbf{X}^{ijh}} \right|_{t=0} \ne \frac{2}{3}$$

The expressions

(5.6) $$(R^{ij})^{-1}_{t=0}, \quad \left| \frac{\mathbf{X}^{ijh} \frac{d}{dt} R^{ij}}{R^{ij} \frac{d}{dt} \mathbf{X}^{ijh}} \right|_{t=0} = \frac{2}{3}$$

determine the movable singularities of the actual solution (1.10).

It is significant to know the rate of convergence of the iterative sequence (7.1-a) relative to the partial sum sequence (7.1-b). The quantity S, in the formulae that follows, measures this rate. It specifies the number of elements in (7.1-b) required to cover the interval between any two elements of the iterative sequence. The quantity S must satisfy the following three inequalities:

(7.12) $$\rho_{\beta+s,\alpha+\tau} \ge \rho_{0,\alpha+\tau} - \rho_{0,\beta+s}, \quad \alpha = 1,2,3,---, \quad \tau = 0,1,2,---$$

(7.13) $$\rho_{\alpha+r,\beta+s+1} \geq \rho_{0,\beta+s+1} - \rho_{0,\alpha+r}$$

(7.14) $$\rho_{0,\beta+s} \leq \rho_{0,\alpha+r} \leq \rho_{0,\beta+s+1}$$

In view of the definition of the metric for a complete metric space,

(7.4) $$\rho_{v,w} \equiv \rho(y_v, y_w) = \max_t \left| y_v(t) - y_w(t) \right| ,$$

the above inequalities are written as

(7.15) $$\max_{t,\iota,h} \left| \bar{X}^{\iota h}_{\beta+s} - X^{\iota h}_{\alpha+r} \right| \geq \max_{t,\iota,h} \left| X^{\iota h}_0 - X^{\iota h}_{\alpha+r} \right| - \max_{t,\iota,h} \left| X^{\iota h}_0 - \bar{X}^{\iota h}_{\beta+s} \right|$$

(7.16) $$\max_{t,\iota,h} \left| X^{\iota h}_{\alpha+r} - \bar{X}^{\iota h}_{\beta+s+1} \right| \geq \max_{t,\iota,h} \left| X^{\iota h}_0 - \bar{X}^{\iota h}_{\beta+s+1} \right| - \max_{t,\iota,h} \left| X^{\iota h}_0 - X^{\iota h}_{\alpha+r} \right|$$

(7.17) $$\max_{t,\iota,h} \left| X^{\iota h}_0 - \bar{X}^{\iota h}_{\beta+s} \right| \leq \max_{t,\iota,h} \left| X^{\iota h}_0 - X^{\iota h}_{\alpha+r} \right| \leq \max_{t,\iota,h} \left| X^{\iota h}_0 - \bar{X}^{\iota h}_{\beta+s+1} \right|$$

The most likely mode in the determination of the quantity S is by a numeric successive approximation process.

Three miscellaneous expressions are given which may prove to be useful. The difference in the error functions over the range of elements from $y^{\iota h}_{\alpha}(t)$ to $y^{\iota h}_{\alpha+r}(t)$ of the iterative sequence, is given by

(7.6) $$\max_{t,\iota h} \left| \epsilon^{\iota h}_{\alpha}(t) - \epsilon^{\iota h}_{\alpha+r}(t) \right| \geq \rho_{\alpha,\alpha+r} - \max_{t,\iota,h} \left| D^{\iota h}_{\alpha+r} - D^{\iota h}_{\alpha} \right|$$

(7.5) $$D^{\iota h}_{\mu} = \frac{A^{\iota}_0}{A^{\iota}_{\mu}} X^{\iota h}_0 + \frac{\sum_{q=0}^{\mu-1} C^{\alpha\,\iota h}_{\mu-q}}{A^{\iota}_{\mu}} .$$

A simple connection between any two elements of the partial sum sequence is given by

(7.8) $$\rho_{\gamma,\gamma+\mu} < \max_{t,\iota,h} \left| \sum_{k=\gamma}^{\gamma+\mu-1} \xi^{\iota h}_{k} \right|$$

The following inequality gives a measure of a mixed metric in terms of consecutive curves (elements) of the partial sum sequence:

$$\rho_{\beta+s,\,\alpha+r} \geq \frac{\rho_{0,\,\beta+s+1} - \rho_{0,\,\beta+s}}{2} \tag{7.18}$$

REFERENCES

1. W. Cheney, Power Series for the N-Body Problem, No. 16 Mathematical Pre-Print Series, Convair Astronautics, Nov. 1958, P.p. 9-10

2. E. L. Ince, Ordinary Differential Equations, pp. 71-72.

3. A. N. Kolmogoroph and S. V. Fomin, Functional Analysis, Vol. 1. Graylock Press, Rochester, N. Y. 1957, pp. 43-44.

4. L. M. Rauch and W. C. Riddel, The Iterative Solutions of the Analytical N-Body Problem, J. Soc. Indust. Appl.Math. Vol. 8, No. 4, Dec. 1960, pp. 568-581.

5. L. M. Rauch, The Explicit Solution of the Analytic N-Body Problem, Submitted for Publication.

6. L. M. Rauch, On An Arbitrary Function in the Iterative Analytic Solution of a General System of Differential Equations, Submitted for Publication.

7. E. T. Whittaker, Analytical Dynamics, Cambridge, 1927, pp.423-424.